Martin Li

Last time ...

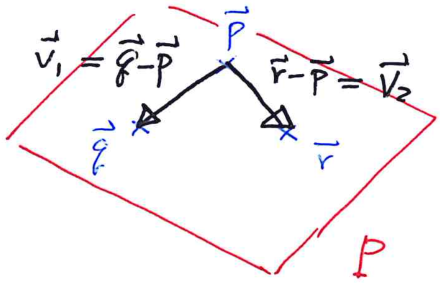
- "$\times$" : cross product in $\mathbb{R}^3$ $\quad \vec{u} \times \vec{v} \in \mathbb{R}^3$
- lines & planes in $\mathbb{R}^2$ or $\mathbb{R}^3$
  - equation form (symmetric)
  - parametric form.

Example: Find a plane P in $\mathbb{R}^3$ passing through the points

$$\vec{p} = (2,2,-3), \quad \vec{q} = (3,5,-1), \quad \vec{r} = (2,0,3).$$

(i) parametric form

(ii) equation form.



Sol: (i) $P = \{ \vec{p} + t\vec{v}_1 + s\vec{v}_2 \mid t, s \in \mathbb{R} \}$

$$\vec{v}_1 = \vec{q} - \vec{p} = (1,3,2)$$
$$\vec{v}_2 = \vec{r} - \vec{p} = (0,-2,6)$$

$$P = \{ (2,2,-3) + t\,(1,3,2) + s\,(0,-2,6) \mid t, s \in \mathbb{R} \}$$

$$\Rightarrow \begin{cases} x = 2 + t \\ y = 2 + 3t - 2s \\ z = -3 + 2t + 6s \end{cases} \quad \text{parametric equations.}$$

(ii) $P: \ ax + by + cz = d$.

1st eq$^{\underline{n}}$ $\Rightarrow$ $t = x - 2$

Plug into 2 eq$^{\underline{n}}$ $\Rightarrow \begin{cases} y = 2 + 3(x-2) - 2s \\ z = -3 + 2(x-2) + 6s \end{cases}$

$$\Rightarrow \quad 3y + z = 11x - 19$$

i.e. $\boxed{11x - 3y - z = 19}$ equation form.

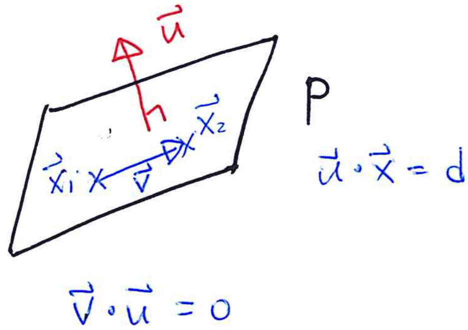
$\vec{u} = (11, -3, -1)$ normal to P

Plane: $ax+by+cz = d$ $\quad P$

$$\Leftrightarrow \quad \underbrace{(a,b,c)}_{\vec{u}} \cdot \underbrace{(x,y,z)}_{\vec{x}} = d \quad \Leftrightarrow \quad \boxed{\vec{u}\cdot\vec{x} = d}$$

vector form

Claim: $\vec{u} \perp P$



Pf: Want to show that

$\vec{u}\cdot(\vec{x}_2 - \vec{x}_1) = 0$ for ANY $\vec{x}_1, \vec{x}_2 \in P$

$$\begin{cases} \vec{x}_1 \in P \Leftrightarrow \vec{u}\cdot\vec{x}_1 = d \\ \vec{x}_2 \in P \Leftrightarrow \vec{u}\cdot\vec{x}_2 = d \end{cases}$$

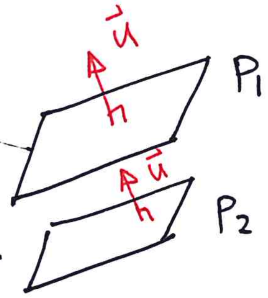
$$\vec{u}\cdot(\vec{x}_2 - \vec{x}_1) = \vec{u}\cdot\vec{x}_2 - \vec{u}\cdot\vec{x}_1 = d - d = 0.$$

※

$ax+by+cz = d_1$

parallel

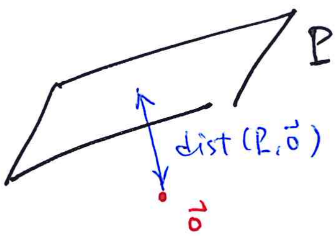
$ax+by+cz = d_2$



Ex: Show that $\dfrac{|d_1 - d_2|}{\sqrt{a^2+b^2+c^2}} = \text{distance}(P_1, P_2)$

Ex: $\{ax+by+cz = d\} = P$

$$\Rightarrow \quad \frac{d}{\sqrt{a^2+b^2+c^2}} = \text{dist}(P, \vec{0})$$

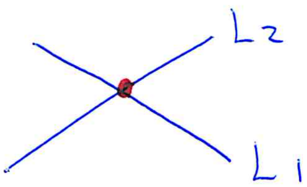


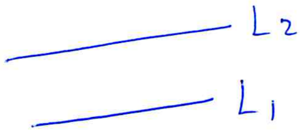
MATH 2010B Advanced Calculus I, Lecture Notes Week 3

Intersections of lines & planes

Martin Li

(1) lines in $\mathbb{R}^2$ (2 eq$^{\underline{n}}$, 2 unknowns)



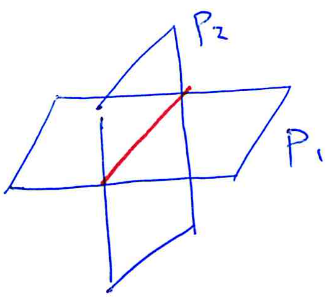


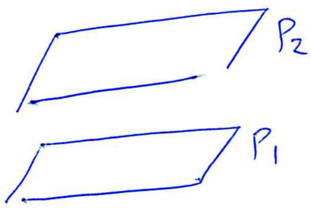
One intersection pt. | NO intersection | ∞ intersection pts

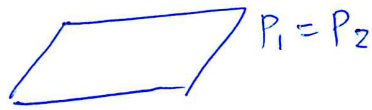
$$\begin{cases} x+y=1 \\ x-y=3 \end{cases} \qquad \begin{cases} x+y=1 \\ x+y=2 \end{cases} \qquad \begin{cases} x+y=1 \\ 2x+2y=2 \end{cases}$$

1 sol$^{\underline{n}}$ | no sol$^{\underline{n}}$ | ∞ sol$^{\underline{n}}$

(2) Planes in $\mathbb{R}^3$ (2 eq$^{\underline{n}}$, 3 unknowns)



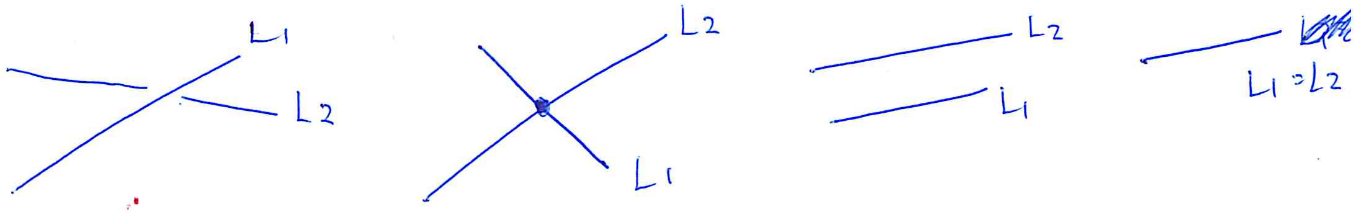




line of intersection | NO intersection | plane of intersection.

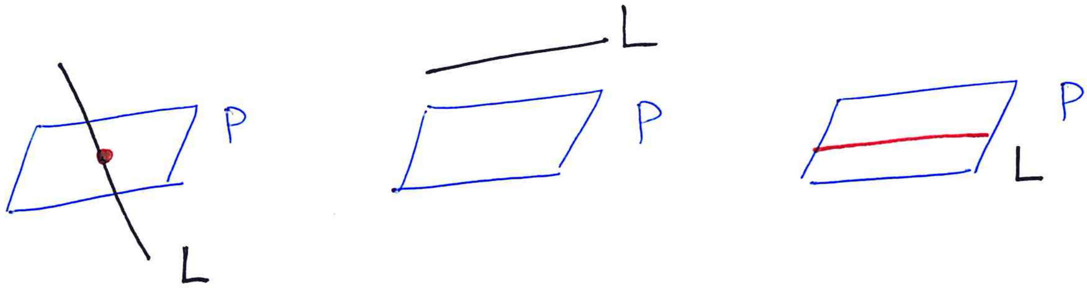
$$\begin{cases} x+y+z=1 \\ x-y+z=2 \end{cases} \qquad \begin{cases} x+y+z=1 \\ x+y+z=2 \end{cases} \qquad \begin{cases} x+y+z=1 \\ 2x+2y+2z=2 \end{cases}$$

(3) lines in $\mathbb{R}^3$



Q: Under this in terms of equations!

(4) line & plane in $\mathbb{R}^3$ (3 eq$^{\underline{n}}$, 3 unknowns)



1 intersection | no intersection | line of intersection

$$\begin{cases} x+y+z=1 & (P) \\ 2x+y+z=2 & \multirow{2}{*}{(L)} \\ x-y-z=3 & \end{cases}$$

Ex:

Ex:

Hyperplane in $\mathbb{R}^n$

$$P = a_1x_1 + a_2x_2 + \cdots + a_nx_n = d$$

$$\boxed{\vec{a}\cdot\vec{x} = d}$$

$$\boxed{\dim P = n-1}$$

(hyperplane)

Last time ...

lines / planes in $\mathbb{R}^2$ or $\mathbb{R}^3$

$$ax + by + cz = d \qquad \text{linear objects}$$

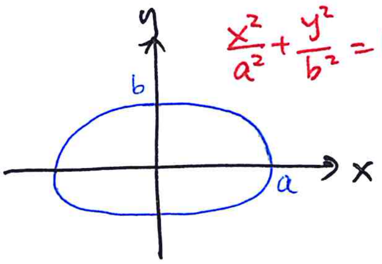
Quadratic Objects in $\mathbb{R}^2$: (x, y)

$$(*) \quad \boxed{\underbrace{Ax^2 + 2Bxy + Cy^2}_{\text{quadratic}} + \underbrace{Dx + Ey}_{\text{linear}} = H}$$

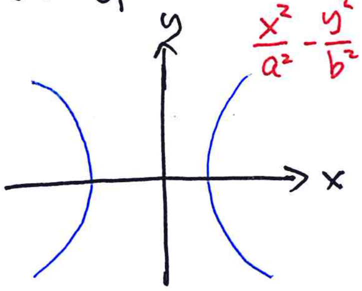
$$\Longleftrightarrow \qquad q(\vec{x}) + \langle \vec{b}, \vec{x} \rangle = H \qquad \vec{x} = (x, y)$$

Example: (Conic Sections)

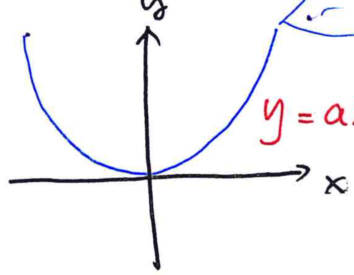
(1) Ellipse $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

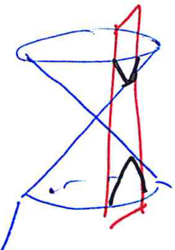


(2) Hyperbola $\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



(3) Parabola $\quad y = ax^2$





Fact: These are all the "nondegenerate" examples, up to change of coordinates.

Q: Given $(*)$, how to determine which conic section is it ?

E.g. Is $x^2 + 2xy + 3y^2 + 4x + 5y = 6$ an ellipse / hyperbola / parabola ?

A: Depends mainly on quadratic part $q(\vec{x})$, using eigenvalues of matrices .....

A Quick Review on Linear Algebra

$m \times n$ matrix $A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \ddots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ columns}} \Big\} m \text{ rows}$

Arithmetics: $A \pm B$, λA componentwise.

Matrix-vector multiplication: A : $m \times n$ matrix.
$\vec{x}$: $n \times 1$ matrix (in $\mathbb{R}^n$)

$$A\vec{x} \in \mathbb{R}^n \qquad \begin{aligned} A : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ \vec{x} &\longmapsto A\vec{x} \end{aligned} \qquad \text{matrix action}$$

E.g. $$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 + 2 \cdot 0 \\ 2 \cdot 1 + 1 \cdot 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Matrix-matrix product: $$\underset{m\times n}{A} \; \underset{n \times k}{B} = \underset{m\times k}{C} \qquad \boxed{AB \neq BA} \;!!$$

match

E.g. $$\underset{2\times 2}{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}} \underset{2\times 2}{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}} = \underset{2\times 2}{\begin{pmatrix} 1 & 3 \\ 2 & 3 \end{pmatrix}}$$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ identity

$AI = A = IA$

Transpose: $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A^t = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

If $A^t = A$, then A is symmetric.

e.g. $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

Eigenvalues A : $n \times n$ square.

$$p(\lambda) := \det(A - \lambda I)$$ (characteristic polynomial of A) (polynomial in λ of degree n)

Set $p(\lambda) = 0 \Rightarrow$ roots $\lambda_1, \dots, \lambda_n$ (eigenvalues of A)

[Recall: $A\vec{x} = \lambda\vec{x} \Rightarrow \vec{x}$ eigenvector w/. eigenvalue λ]

Example: Find the eigenvalues of $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Sol:
$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$p(\lambda) := \det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = (-\lambda)^2 - 1^2 = \lambda^2 - 1$$

Set $p(\lambda) = \lambda^2 - 1 = 0 \Rightarrow$ roots $\lambda_1 = -1$, $\lambda_2 = 1$. #

Ex: Do it for $A = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$.

Back to quadratic objects ooo ooo.

$$q(x,y) = Ax^2 + 2Bxy + Cy^2$$ quadratic form in $\mathbb{R}^2$

$$\overset{(Ex:)}{=} (x \;\; y) \begin{pmatrix} A & B \\ B & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \vec{x}^t M \vec{x}$$

$1 \times 2 \quad 2 \times 2 \quad 2 \times 1$

$M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ symmetric $M^t = M$

$$\left\{ \begin{array}{c} \text{quadratic forms in } \mathbb{R}^2 \\ q(x,y) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} 2 \times 2 \text{ symmetric matrices} \\ M \end{array} \right\}$$

Theorem (Spectral Thm).

The eigenvalues of an $n \times n$ symmetric matrices are all real numbers.

i.e. $A^t = A \implies \lambda_1, \dots, \lambda_n \in \mathbb{R}$.

Idea: $q(x,y) = \vec{x}^T M \vec{x} \rightsquigarrow M$ symmetric $\rightsquigarrow \lambda_1, \lambda_2 \in \mathbb{R}$

" The shape of $q(\vec{x}) + \langle \vec{b}, \vec{x} \rangle = H$ is determined by the relative signs of λ_1, λ_2."

$$M = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$q(x,y) = \lambda_1 x^2 + \lambda_2 y^2$$

Thm:
$$\begin{cases} \lambda_1 \lambda_2 > 0 \ (\text{i.e. same sign}) \implies \text{ellipse}. \\ \lambda_1 \lambda_2 < 0 \ (\text{i.e. diff. sign}) \implies \text{hyperbola} \\ \lambda_2 = 0 \implies \text{parabola} \end{cases}$$

in the non-degenerate cases.

E.g. $\underbrace{x^2 + 2xy + 3y^2}_{q(x,y)} + 4x + 5y = 6$

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$p(\lambda) = \det(M - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}$$

$$= \lambda^2 - 4\lambda + 2 = 0$$

$$\implies \lambda = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$$

$\lambda_1 = 2 + \sqrt{2}$ and $\lambda_2 = 2 - \sqrt{2}$

so $\lambda_1 > 0$, $\lambda_2 > 0$

$\implies$ ellipse!. (why?)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Change of coordinates

$$x^2+2xy+3y^2+4x+5y = 6$$

$$(x^2+2xy+y^2)+(\sqrt{2}y)^2+4x+5y=6$$

complete the square

$$\underbrace{(x+y)}_{u}^2+\underbrace{(\sqrt{2}y)}_{v}^2+4\underbrace{(x+y)}_{u}+\frac{1}{\sqrt{2}}\underbrace{(\sqrt{2}y)}_{v} = 6$$

$$u^2+v^2+4u+\frac{1}{\sqrt{2}}v = 6$$

$$(u+2)^2+\left(v+\frac{1}{2\sqrt{2}}\right)^2 = 6+4+\frac{1}{8} = \frac{81}{8}.$$

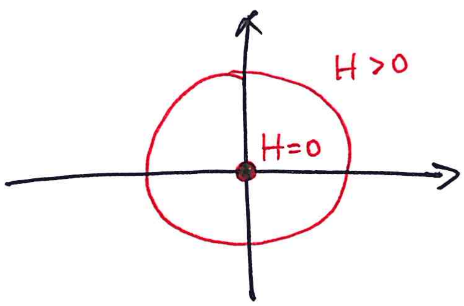
$$\underbrace{(u+2)}_{\xi}^2+\underbrace{\left(v+\frac{1}{2\sqrt{2}}\right)}_{\eta}^2 = \frac{81}{8}$$

circle.

Q: How to do this systematically?

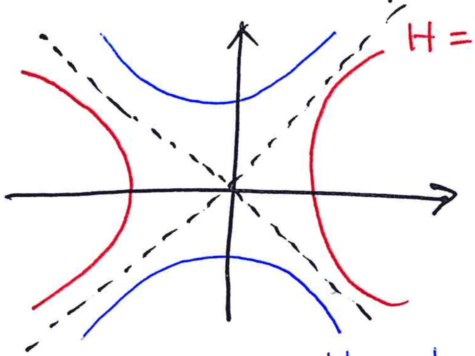
Degeneracy

(i) $x^2+y^2 = H$



H<0: empty.

(ii) $x^2-y^2 = H$



Quadric surfaces in $\mathbb{R}^3$: (x, y, z)

$$\overbrace{Ax^2 + By^2 + Cz^2 + 2Pxy + 2Qyz + 2Rxz}^{q(x,y,z) \text{ quadratic}} + \underbrace{Dx + Ey + Fz}_{\text{linear } \langle \vec{b}, \vec{x} \rangle} = H$$

$$q(x,y,z) = (x \ \ y \ \ z) \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} A & P & R \\ P & B & Q \\ R & Q & C \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\| \quad M \quad 3 \times 3$ symmetric matrix.

$\Downarrow$ (Spectral)

$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$

Relative signs of $\lambda_1, \lambda_2, \lambda_3$

Case 1: $\lambda_3 = 0$

- $\lambda_1 \lambda_2 > 0$
- $\lambda_1 \lambda_2 < 0$
- $\lambda_2 = 0$.

Case 2: $\lambda_3 \neq 0$ (all $\lambda_1, \lambda_2 \neq 0$)

- all same sign.
- 1 is different sign.

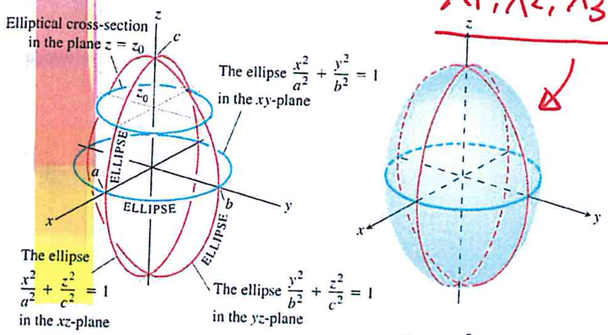
λ_1	λ_2	λ_3
+	+	+
+	+	−
+	−	−
−	−	−

1 is different

all same sign

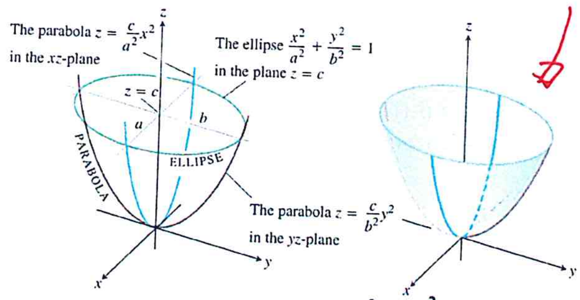
TABLE 12.1 Graphs of Quadric Surfaces

$\lambda_1, \lambda_2, \lambda_3 > 0$

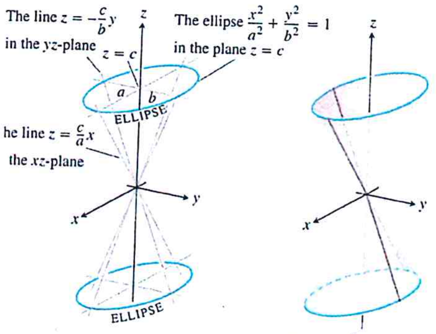


ELLIPSOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$\lambda_1, \lambda_2 > 0$, $\lambda_3 = 0$.

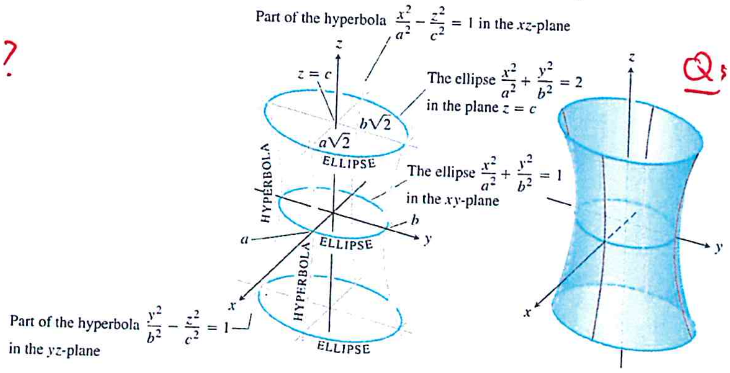


ELLIPTICAL PARABOLOID $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$



Q: ?

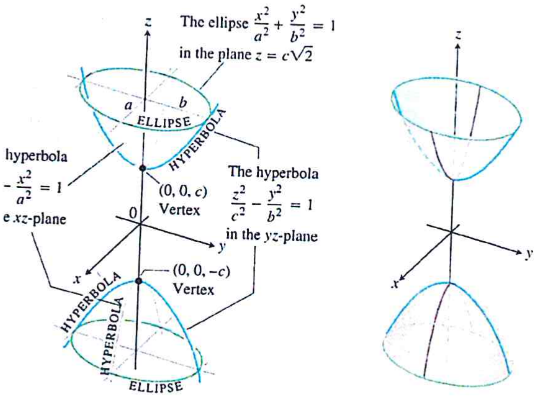
ELLIPTICAL CONE $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$



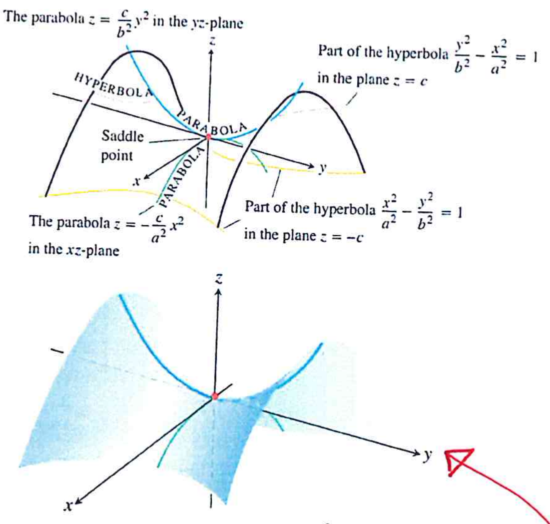
Q: ?

HYPERBOLOID OF ONE SHEET $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

Q: ?



YPERBOLOID OF TWO SHEETS $\quad \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



HYPERBOLIC PARABOLOID $\quad \frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$

$\lambda_1 \lambda_2 < 0$, $\lambda_3 = 0$

Sources:
Thomas' Calculus by Thomas, Weir and Hass, of Pearson.